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Best Polynomial Approximation to Certain Entire Functions

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INTRODUCTION

Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$ and let

$$E_n(f) = \inf_{P \in \pi_n} \|f - P\|_{\mathscr{L}^\infty[-1,1]}, \qquad n = 0, 1, 2,..., \tag{1}$$

be the minimum error in the Chebyshev approximation of $f(x)$ over the set π_n of real polynomials of degree at most n. Bernstein [1, p. 118] proved that

$$\lim_{n \to \infty} [E_n(f)]^{1/n} = 0 \tag{2}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and let

$$M(r) = \max_{|z|=r} |f(z)|;$$

then the order ρ and lower order λ of $f(z)$ are defined as [2, p. 8]:

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r)}{\log r} = \begin{matrix} \rho \\ \lambda \end{matrix} \qquad (0 \leqslant \lambda \leqslant \rho \leqslant \infty). \tag{3}$$

An entire function $f(z)$ is said to be of regular growth if $\rho = \lambda$. For $0 < \rho < \infty$, the type T and lower type t of $f(z)$ are defined by

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix} T \\ t \end{matrix} \qquad (0 \leqslant t \leqslant T \leqslant \infty). \tag{4}$$

An entire function $f(z)$ is said to be of perfectly regular growth if $0 < t =$

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$T < \infty$. For entire functions of zero order, following Shah and Ishaq [8], we define the logarithmic order ρ^* and lower logarithmic order λ^* by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log\log r} = \begin{matrix}\rho^*\\ \lambda^*\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \rho^* \leqslant \infty). \tag{5}$$

Further, for $1 < \rho^* < \infty$, the logarithmic type T^* and lower logarithmic type t^* are defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix} \qquad (0 \leqslant t^* \leqslant T^* \leqslant \infty). \tag{6}$$

An entire function $f(z)$ is said to be of regular logarithmic growth if $\rho^* = \lambda^*$ and of perfectly regular logarithmic growth if $1 < \rho^* < \infty$ and $0 < t^* = T^* < \infty$.

Varga [10, Theorem 1] proved that

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log[E_n(f)]^{-1}} = \rho \tag{7}$$

satisfies $0 \leqslant \rho < \infty$ if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ. Later, Reddy [5, 6], Juneja [4], etc., obtained some relations between the rate of decrease of $E_n(f)$ and $|a_n|$ for entire functions of finite, zero or infinite order.

S. M. Shah ([7], see also an earlier paper by Seremeta referred to in [7]) introduced the notion of generalized order of entire functions, which includes all classes of entire functions. Thus, if we put $\alpha(x) = \log x$, $\beta(x) = x$ in (1.3) [7, p. 316], we get the definitions of order and lower order, while if we substitute $\alpha(x) = \beta(x) = \log x$, then we get the definitions of logarithmic and lower logarithmic order. Shah also proved a result [7, Theorem 3] which extends Varga's result mentioned above as well as some theorems of Reddy [5, Theorems 1, 2A, 2B]. Recently, A. Giroux [3] considered the approximation of entire functions on bounded domains in the complex plane. His main result [3, Theorem] also extends Varga's result as well as a result of Reddy [5, Theorem 3].

Reddy [6, Theorems 6, 7, 11 and 12] obtained some asymptotic relations between the Taylor coefficients a_n and $E_n(f)$ for entire functions of regular growth and perfectly regular growth, which are based on the following results of Valiron [9, pp. 41–45]:

THEOREM A. *A necessary and sufficient condition that an entire function f be of regular growth is that the coefficients a_n's satisfy, for every $\varepsilon > 0$, the inequality*

$$|a_n|^{1/n} < n^{-1/(\rho+\varepsilon)}, \qquad \textit{for all large } n, \tag{8}$$

and that there exists a strictly increasing sequence $\{n_p\}_1^\infty$ *of positive integers such that*

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1$$

and

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log |a_{n_p}|^{-1}} = \rho. \tag{9}$$

THEOREM B. *An entire function* f *is of perfectly regular growth* (ρ, T), $0 < \rho < \infty$, $0 < T < \infty$, *if and only if, given* $\varepsilon > 0$, *there exists an* $n_0(\varepsilon)$ *such that*

$$\frac{n}{e\rho} |a_n|^{\rho/n} < T + \varepsilon, \qquad \textit{for} \quad n > n_0(\varepsilon), \tag{10}$$

and there exists a strictly increasing sequence $\{n_p\}_1^\infty$ *of positive integers such that*

$$\lim_{p\to\infty} \frac{n_{p+1}}{n_p} = 1$$

and

$$\lim_{p\to\infty} \frac{n_p}{e\rho} |a_{n_p}|^{\rho/n_p} = T. \tag{11}$$

The proofs of Theorems 6, 7, 11 and 12 in [7] are based on the wrong assumption that two entire functions of regular growth and same order will have same sequence $\{n_p\}$ of positive integers satisfying (9). Further, in Theorems 7, 11 and 12, it is presumed that for entire functions of regular or perfectly regular growth, the limit

$$\lim_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} \qquad \text{or} \qquad \lim_{n\to\infty} n\,|a_n|^{\rho/n}$$

exists, respectively. This is also not always true. To see this, let us consider two entire functions

$$\cos z = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)!,$$

both of which are of perfectly regular growth, order 1 and type 1. It is clear from Valiron's proof of Theorem A [9, p. 41] that $n_p = 2p$ for $\cos z$ while $n_p = 2p+1$ for $\sin z$. Further, for $\cos z$,

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log |a_n|^{-1}} = 1, \qquad \lim_{n\to\infty} \inf \frac{n \log n}{\log |a_n|^{-1}} = 0, \qquad \text{etc.}$$

It is the aim of this paper to prove the above-mentioned results of Reddy [6] under an additional condition on E_n's, which yields better results. The same type of results for entire functions of zero order have also been obtained.

Results

We now prove our results. We shall assume throughout that the entire functions considered have real coefficients in their Taylor series expansions.

THEOREM 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of positive order and regular growth. Then there exists a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers such that*

$$\lim_{p\to\infty} \frac{\log E_{n_p}}{\log |a_{n_p}|} = 1 \tag{12}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n > n_0$.

Proof. Let $f(z)$ be of order $\rho > 0$. Since $E_n(f)/E_{n+1}(f)$ is a non-decreasing function of n for $n > n_0$, we get from Theorem 3 of Shah [7], on substituting $\alpha(n) = \log n$, $\beta(n) = n$,

$$\lim_{n\to\infty} \frac{n \log n}{\log [E_n(f)]^{-1}} = \rho. \tag{13}$$

Further, from Theorem A stated before, there exists a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers such that

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log |a_{n_p}|^{-1}} = \rho. \tag{14}$$

Hence, for given $\varepsilon > 0$, we get from (13) and (14) the following inequalities for $p > p_0(\varepsilon)$:

$$\frac{n_p \log n_p}{\rho + \varepsilon} < \log [E_{n_p}(f)]^{-1} < \frac{n_p \log n_p}{\rho - \varepsilon},$$

$$\frac{n_p \log n_p}{\rho + \varepsilon} < \log |a_{n_p}|^{-1} < \frac{n_p \log n_p}{\rho - \varepsilon}.$$

Hence, we have, for $p > p_0(\varepsilon)$,

$$\frac{\rho-\varepsilon}{\rho+\varepsilon} < \frac{\log[E_{n_p}(f)]^{-1}}{\log|a_{n_p}|^{-1}} < \frac{\rho+\varepsilon}{\rho-\varepsilon},$$

which leads to (12) on taking limits as $p \to \infty$. This proves Theorem 1.

THEOREM 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of perfectly regular growth (ρ, T). Then there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that*

$$\lim_{p\to\infty} [E_{n_p}(f)/|a_{n_p}|]^{1/n_p} = 1/2 \tag{15}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n > n_0$.

Proof. Consider the entire function

$$H(\sigma) = \sum_{n=0}^{\infty} E_n(f)\,\sigma^n$$

which is also of order ρ and type $T/2^{\rho}$ (see [5, Eqs. (16), (18) and (22)]). Under the given condition on E_n's, we have from Theorem 3 and Lemmas 4A and 4B of Reddy [5] that

$$\lim_{n\to\infty} n[E_n(f)]^{\rho/n} = e\rho T/2^{\rho}. \tag{16}$$

From Theorem B, there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that

$$\lim_{p\to\infty} n_p\,|a_{n_p}|^{\rho/n_p} = e\rho T. \tag{17}$$

Proceeding as in Theorem 1, we get (15) from (16) and (17). This proves Theorem 2.

In the next two theorems, we consider the asymptotic relations between minimum Chebyshev errors of two entire functions having same order and same type.

THEOREM 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two entire functions of same positive order ρ and regular growth. Then there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that*

$$\lim_{p\to\infty} [\log E_{n_p}(f)/\log E_{n_p}(g)] = 1 \tag{18}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n > n_0$

(alternatively, $E_n(g)/E_{n+1}(g)$ forms a non-decreasing function of n for $n > n_0$).

Proof. Let us consider the two entire functions

$$H(\sigma) = \sum_{n=0}^{\infty} E_n(f)\,\sigma^n, \qquad G(\sigma) = \sum_{n=0}^{\infty} E_n(g)\,\sigma^n.$$

Then $H(\sigma)$ and $G(\sigma)$ are also of same order ρ and regular growth (see [5, Eqs. (16) and (18)]). Again, from [7, Theorem 3], we have

$$\lim_{n\to\infty} \frac{n \log n}{\log[E_n(f)]^{-1}} = \rho. \tag{19}$$

From Theorem A, a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers exists for which

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log[E_{n_p}(g)]^{-1}} = \rho. \tag{20}$$

Theorem 3 now follows from (19) and (20).

THEOREM 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two entire functions of same perfectly regular growth (ρ, T). Then there exists a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers such that*

$$\lim_{p\to\infty} [E_{n_p}(f)/E_{n_p}(g)]^{1/n_p} = 1 \tag{21}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n > n_0$.

Proof. Since $f(z)$ and $g(z)$ are entire functions of perfectly regular growth (ρ, T), $H(\sigma)$ and $G(\sigma)$, as defined before, are also entire functions of perfectly regular growth $(\rho, T/2^\rho)$. From Theorem B, there exists a strictly increasing sequence $\{n_p\}$ of positive integers such that

$$\lim_{p\to\infty} n_p[E_{n_p}(g)]^{\rho/n_p} = e\rho T/2^\rho. \tag{22}$$

The result now follows on combining (16) and (22).

Remark 1. As already mentioned, the proofs of Theorems 6, 7, 11 and 12 of [6] are based on the assumption that relations (9) and (11) hold simultaneously for two entire functions of same order and same type (see [6, Eqs. (21) and (24)]). That this is not always true is clear from the example of two entire functions $\cos z$ and $\sin z$. Hence, to complete the proofs of these theorems, we have to take the additional condition on $E_n(f)$ in our

Theorems 1 to 4. The additional condition imposed thus gives the strictly increasing sequence $\{n_p\}$ of positive integers in place of the sequence $\{n_p\}$ with $n_p \to \infty$ as $p \to \infty$, as given in Theorems 7, 11 and 12 [6].

We now give some asymptotic relations between a_n and $E_n(f)$ for entire functions of zero order. We shall be making use of the following results, which can be easily derived on the lines of Valiron [9, pp. 41–45]:

(i) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of logarithmic order ρ^* and lower logarithmic order λ^*, $1 < \lambda^* \leqslant \rho^* < \infty$, then for $\varepsilon > 0$ and arbitrarily small,

$$|a_n|^{1/n} < \exp[-n^{1/(\rho^* - 1 + \varepsilon)}] \tag{23}$$

for all sufficiently large values of n. Further, there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that

$$\limsup_{p \to \infty} \frac{\log n_{p+1}}{\log n_p} \leqslant \frac{\rho^* - 1}{\lambda^* - 1} \tag{24}$$

and

$$|a_{n_p}|^{1/n_p} > \exp[-(n_p)^{1/(\lambda^* - 1 - \varepsilon)}]. \tag{25}$$

(ii) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of logarithmic order ρ^* $(1 < \rho^* < \infty)$, logarithmic type T^* and lower logarithmic type t^* $(0 < t^* \leqslant T^* < \infty)$, then, given arbitrarily small $\varepsilon > 0$,

$$|a_n|^{1/n} < \exp\left[-\frac{(\rho^* - 1)}{\rho^*}\left\{\frac{n}{\rho^* T^* + \varepsilon}\right\}^{1/(\rho^* - 1)}\right] \tag{26}$$

for all large values of n. Further, there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers satisfying

$$\limsup_{p \to \infty} \frac{n_{p+1}}{n_p} \leqslant \frac{x_2}{x_1} \tag{27}$$

for which

$$|a_{n_p}|^{1/n_p} > \exp\left[-\left(\frac{\rho^* - 1}{\rho^*}\right)\left\{\frac{n_p}{\rho^* t^* - \varepsilon}\right\}^{1/(\rho^* - 1)}\right], \tag{28}$$

where x_1 and x_2 are the smallest and largest roots of the equation

$$(\rho^* - 1)\, x^{\rho^*} - \rho^* x^{(\rho^* - 1)} + t^*/T^* = 0. \tag{29}$$

We now prove

THEOREM 5. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of logarithmic order ρ^* and lower logarithmic order λ^*, $1<\lambda^* \leqslant \rho^* < \infty$. Then there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that*

$$1-\frac{(\rho^*-\lambda^*)}{\lambda^*(\rho^*-1)} \leqslant \lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log[E_{n_p}(f)]^{-1}}{\log\log|a_{n_p}|^{-1}} \leqslant 1+\frac{(\rho^*-\lambda^*)}{\rho^*(\lambda^*-1)} \tag{30}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$.

Proof. Since $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$, we have, from [5, Theorems 5, 6A and 6B],

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n}{\log[(1/n)\log\{E_n(f)\}^{-1}]} = \frac{\rho^*-1}{\lambda^*-1}.$$

Hence we have for arbitrarily small $\varepsilon>0$ and all $n>n_0(\varepsilon)$,

$$n^{(\rho^*+\varepsilon)/(\rho^*+\varepsilon-1)} < \log[E_n(f)]^{-1} < n^{(\lambda^*-\varepsilon)/(\lambda^*-\varepsilon-1)}. \tag{31}$$

From (23) and (25), we have for the sequence $\{n_p\}$ satisfying (24),

$$(n_p)^{(\rho^*+\varepsilon)/(\rho^*+\varepsilon-1)} < \log|a_{n_p}|^{-1} < (n_p)^{(\lambda^*-\varepsilon)/(\lambda^*-\varepsilon-1)}. \tag{32}$$

Hence from (31) and (32), we have

$$\frac{(\rho^*+\varepsilon)(\lambda^*-1-\varepsilon)}{(\lambda^*-\varepsilon)(\rho^*-1+\varepsilon)} < \frac{\log\log[E_{n_p}(f)]^{-1}}{\log\log|a_{n_p}|^{-1}} < \frac{(\lambda^*-\varepsilon)(\rho^*+\varepsilon-1)}{(\rho^*+\varepsilon)(\lambda^*-\varepsilon-1)}.$$

The result now follows on proceeding to limits as $p\to\infty$. This proves Theorem 5.

COROLLARY 1. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of regular logarithmic growth and logarithmic order ρ^* $(1<\rho^*<\infty)$. Then, there exists a strictly increasing sequence of positive integers $\{n_p\}_1^{\infty}$, such that*

$$\lim_{p\to\infty} \frac{\log\log[E_{n_p}(f)]^{-1}}{\log\log|a_{n_p}|^{-1}} = 1 \tag{33}$$

and

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1 \tag{34}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$.

This follows immediately on taking $\rho^*=\lambda^*$ in the inequalities (23) to (25) and (30).

Next we prove

THEOREM 6. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of logarithmic order ρ^*, logarithmic type T^* and lower logarithmic type t^*, $0<t^*\leqslant T^*<\infty$. Then there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers, such that*

$$\left(\frac{t^*}{T^*}\right)^{1/\rho^*-1} \leqslant \lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[E_{n_p}(f)]^{-1}}{\log|a_{n_p}|^{-1}} \leqslant \left(\frac{T^*}{t^*}\right)^{1/\rho^*-1} \tag{35}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$.

The result follows on using Theorems 7, 8 and 9 of [5] together with (26) and (28). Hence we omit the proof.

From Theorem 2, we get, on putting $T^*=t^*$,

COROLLARY 2. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of perfectly regular logarithmic growth (ρ^*, T^*), $0<T^*<\infty$, such that $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$. Then, for a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers, we have*

$$\lim_{p\to\infty} \frac{\log E_{n_p}(f)}{\log|a_{n_p}|}=1 \tag{36}$$

such that

$$\lim_{p\to\infty} \frac{n_{p+1}}{n_p}=1. \tag{37}$$

Remark 2. Reddy [6, Theorem 10] proved a somewhat better result than (35) but under the additional condition that $|a_n/a_{n+1}|$ also forms a non-decreasing function of n for $n>n_0$.

Remark 3. The above Corollary 2 is an extension of Theorem 1 for entire functions of zero order.

In the end, we give two theorems, which give asymptotic relations between the minimum Chebyshev errors of two entire functions of same logarithmic growth.

THEOREM 7. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ be two entire functions of regular logarithmic growth and logarithmic order ρ^*, $1<\rho^*<\infty$. Then there exists a strictly increasing sequence $\{n_p\}_1^{\infty}$ of positive integers such that*

$$\lim_{p\to\infty} \frac{\log\log[E_{n_p}(f)]^{-1}}{\log\log[E_{n_p}(g)]^{-1}}=1 \tag{38}$$

provided $E_n(f)/E_{n+1}(f)$ forms a non-decreasing function of n for $n>n_0$.

The proof of the above theorem is similar to that of Theorem 3. Here we use Lemmas 5, 6A and 6B of [5] together with the inequalities (23) and (25) when $\rho^* = \lambda^*$.

THEOREM 8. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two entire functions of perfectly regular logarithmic growth* (ρ^*, T^*). *Then there exists a strictly increasing sequence* $\{n_p\}_1^{\infty}$ *of positive integers such that*

$$\lim_{p \to \infty} \frac{\log E_{n_p}(f)}{\log E_{n_p}(g)} = 1 \tag{39}$$

provided $E_n(f)/E_{n+1}(f)$ *forms a non-decreasing function of* n *for* $n > n_0$.

This follows on using the inequalities (26) and (29) for $T^* = t^*$ and Theorems 7, 8 and 9 of [5].

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